JOURNAL OF ALGEBRA 47, 433–440 (1977)

Doubly Transitive Permutation Groups in Which the One-Point Stabilizer is Triply Transitive on a Set of Blocks

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Communicated by Walter Feit

Received May 3, 1976

Suppose that G is a doubly transitive permutation group on a finite set Ω and that for α in Ω the stabilizer G_α has a set $\Sigma = \{B_1, ..., B_t\}$ of nontrivial blocks of of imprimitivity in $\Omega - \{\alpha\}$, that is, $|B_i| > 1$ and $|\Sigma| > 1$. In two previous papers [5, 6] it was shown that apart from a few known groups, the setwise stabilizer of B_1 in G_α acts faithfully on B_1 if $G_\alpha{}^\Sigma$ is the alternating or symmetric group, one of the Mathieu groups, or a normal extension of $PSL(2, q)$ in their usual representations. The raises the question:

QUESTION. If $G_\alpha{}^\Sigma$ is multiply transitive, is it possible to characterize the groups G for which the setwise stablizer in G_α of the block B of Σ does not act faithfully on B?

The only groups I know of in which G_α is 2-transitive on Σ and for B in Σ, the stablizer of B is not faithful on B, are the following:

(i) $PSL(n, q) \leqslant G \leqslant P\Gamma L(n, q)$, for $n \geqslant 3$ in its natural representation. If $B \in \Sigma$ then $B \cup \{\alpha\}$ is a line and $PSL(n-1, q) \leqslant G_\alpha{}^\Sigma \leqslant P\Gamma L(n-1, q)$.

(ii) $PSU(3, q) \leqslant G \leqslant P\Gamma U(3, q)$ permuting the set of absolute points of the projective plane over a field of q^2 elements. If $B \in \Sigma$ then $B \cup \{\alpha\}$ is a non-absolute line, $|B| = q$, $|\Sigma| = q^2$, and $G_\alpha{}^\Sigma$ has a regular normal subgroup of order q^2.

(iii) $G = A_7$ acting on the 15 points of the projective geometry of dimension 3 over a field of two elements; $G_\alpha \simeq PSL(2, 7)$ acts 2-transitively on the set of lines containing α.

(iv) G has a regular normal subgroup.

If we drop the assumption that $G_\alpha{}^\Sigma$ is 2-transitive, then we have another family of examples.

(v) A group G of Ree type $R(q)$ is 2-transitive on $q^3 + 1$ points. For any two distinct points α, β, there is a unique nontrivial element g in $G_{\alpha\beta}$ which

ISSN 0021-8693

fixes at least three points. The set of points distinct from α which are fixed by g is a block of imprimitivity for G_α which is fixed pointwise by g.

The aim of this paper is to give an affirmative answer to the question in the case where G_α is 3-transitive on Σ, but not faithful on Σ. Different methods will be needed to deal with the case G_α faithful on Σ. Throughout the paper we assume the following hypothesis.

HYPOTHESIS (*). (a) *G is a doubly transitive permutation group on a set Ω of n points. For α in Ω the stabilizer G_α has a set $\Sigma = \{B_1, ..., B_t\}$ of nontrivial blocks of imprimitivity, where $|\Sigma| = t > 1$, $|B_i| = b > 1$, and $n = 1 + tb$.*

(b) *We denote by K_i the subgroup of G_α fixing B_i setwise, by $\bar{K}_i$ the subgroup fixing B_i pointwise, and by H the subgroup fixing all blocks of Σ setwise.*

(c) $H \neq 1, \bar{K}_i \neq 1$.

The result of this paper, which is stated below, is a generalization of [5, Theorems 1 and 2; 6].

THEOREM. *Assume Hypothesis (*) and assume that G_α^Σ is 3-transitive of degree $t \geqslant 3$. Then $PGL(2, q) \leqslant G_\alpha^\Sigma \leqslant P\Gamma L(2, q)$ in its natural representation where $q = t - 1$ is a prime power. Moreover, either*

(a) *$PSL(3, q) \leqslant G \leqslant P\Gamma L(3, q)$ in its natural representation, or*

(b) *Ω with the translates under G of $B_1 \cup \{\alpha\}$ as lines is an affine translation plane of order q, and G contains the translation group.*

An immediate corollary of this result which follows from [5] is:

COROLLARY. *If Hypothesis (*) is true and if G_α^Σ is 4 transitive, then $G_\alpha^\Sigma \simeq S_t$ where t is 4 or 5, and either*

(a) *$PSL(3, t-1) \leqslant G \leqslant P\Gamma L(3, t-1)$ in its natural representation, or*

(b) *$t = 4$, G has a regular elementary abelian normal subgroup of order 9, and H is cyclic of order 2.*

Notation. Most of the notation used follows the conventions of [1, 7]. If a group G has a permutation representation on a set Σ, the constituent of G on Σ is denoted by G^Σ; the set of fixed points of G in Σ is denoted by $\mathrm{fix}_\Sigma\, G$ and simply by fix G if the set Σ is clear from the context; and orbits of G containing more than one point are called long G-orbits.

DEFINITION. A block design consists of a set of v points and a set of b blocks with a relation called incidence between points and blocks, such that any block is incident with k points and any two points with λ blocks, where $\lambda > 0$ and

$2 \leqslant k < v - 1$. The number r of blocks incident with a given point is also constant. If $k > 2$ it is called a proper design. By easy counting arguments we have

$$vr = bk, \qquad v(v-1)\lambda = bk(k-1).$$

Also it is well known that $b \geqslant v$, or equivalently $r \geqslant k$.

Proof of the Theorem. Assume that Hypothesis (*) is true and that G_α^Σ is 3-transitive of degree $t \geqslant 3$. It is easy to check that a group G satisfying $PSL(mq) \leqslant G \leqslant P\Gamma L(m, q)$ in its natural representation for some $m \geqslant 3$ and prime power q, satisfies Hypothesis (*) where $\{B_i \cup \{\alpha\} \mid i = 1,..., t\}$ is the set of lines containing α, and these B_i are the only nontrivial blocks for G_α. Further, since $PGL(m-1, q) \leqslant G_\alpha^\Sigma \leqslant P\Gamma L(m-1, q)$ in its natural representation, then G_α^Σ is 3-transitive if and only if $m = 3$. Thus in this case the theorem is true. So we assume that G does not satisfy $PSL(m, q) \leqslant G \leqslant P\Gamma L(m, q)$ for any $m \geqslant 3$. Then by [6, Lemma 1.1], H is semiregular on $\Omega - \{\alpha\}$ and G is a group of automorphisms of a block design with $\lambda = 1$, the blocks of which are the translates under G of the set $B_1 \cup \{\alpha\}$.

PROPOSITION 1. *If K_1 has a normal subgroup which acts regularly on $\Sigma - \{B_1\}$ then the theorem is true.*

Proof. Assume that K_1 has a normal subgroup which is regular on $\Sigma - \{B_1\}$. Then by [2, Theorem 1.1], the fact that G_α^Σ is 3-transitive, and [7, 10.2 and 11.3], either

(a) $PGL(2, q) \leqslant G_\alpha^\Sigma \leqslant P\Gamma L(2, q)$ in its natural representation where $q = t - 1$ is a prime power, or

(b) G_α^Σ has a regular elementary abelian normal subgroup of order t, where $t = 3$ or $t = 2^a$ for some integer $a \geqslant 2$ (since $t \geqslant 3$).

In case (a) the theorem is true by [6]. If $t = 3$, the theorem is true by [5, Theorem 1], for $G_\alpha^\Sigma = S_3 \simeq PSL(2, 2)$. So suppose that G_α^Σ has a regular normal subgroup which is elementary abelian of order $t = 2^a$ for some $a \geqslant 2$. Then G_α has a normal subgroup N containing H such that $N^\Sigma = N/H$ is elementary abelian and regular, and N^Σ is the unique minimal normal subgroup of G_α^Σ, (by [7, 11.5, 10.1]).

Let $M = \langle \bar{K}_i \mid i = 1,..., t\rangle$. Since each $\bar{K}_i$ centralizes H, (for H and $\bar{K}_i$ are normal subgroups of K_i with trivial intersection), then also M centralizes H. If $M \cap H$ is nontrivial, then it is an abelian normal subgroup of G_α. On the other hand, if $M \cap H$ is trivial, then M is isomorphic to M^Σ, a nontrivial normal subgroup of G_α^Σ. Hence M has a subgroup which is normal in G_α, elementary abelian of order 2^a, and acts regularly on Σ. Thus in either case G_α has a nontrivial abelian normal subgroup, and since the degree $n = 1 + 2^a b$ is odd, it follows from [4, Theorem B] that G is a normal extension of one of

(a) $PSL(m, q)$ where $(q^m - 1)/(q - 1)$ is odd and $m \geqslant 2$,

(b) $PSU(3, q)$,

(c) $S_z(q)$, in their natural representations, or

(d) G has a regular normal subgroup.

We are assuming that G does not satisfy (a) if $m \geqslant 3$, and $PSL(2, q)$ is 2-primitive and so cannot satisfy Hypothesis (*). By considering the possible parameters of a block design, it is easy to show that $S_z(q)$ cannot act as a group of automorphisms of a block design with $\lambda = 1$. Also no normal extension of $PSU(3, q)$ can satisfy Hypothesis (*) with G_α^Σ 3-transitive. Hence G has a regular normal subgroup, R, say.

Consider the group NR, where N is the normal subgroup of G_α containing H such that $N^\Sigma = N/H$ is elementary abelian and regular. Since H is semiregular on $\Omega - \{\alpha\}$ and N is regular on Σ it is easy to see that N is semiregular on $\Omega - \{\alpha\}$. Thus NR is a Frobenius group. By [1, 10.3.1], the Sylow 2-subgroups of N are cyclic or generalized quaternion, and since N/H is elementary abelian of order $2^a \geqslant 4$, it follows that $2^a = 4$. Thus $G_\alpha^\Sigma \simeq S_4 \simeq PGL(2, 3)$, and the theorem follows in this case from [5, Theorem 1] or [6]. This completes the proof of Proposition 1.

Thus we may assume that K_1 has no normal subgroup acting regularly on $\Sigma - \{B_1\}$.

LEMMA 2. (a) *$\bar{K}_1 \cap K_2$ fixes the blocks B_1 and B_2 setwise and has nontrivial orbits of equal length in $\Sigma - \{B_1, B_2\}$.*

(b) *The centralizer $C_G(\bar{K}_1)$ of $\bar{K}_1$ acts as a Frobenius group on $B_1 \cup \{\alpha\}$ of degree $1 + b = r^c$ for some prime r and positive integer c. Moreover, $C_G(\bar{K}_1)$ has a normal subgroup A which is elementary abelian of order r^c and acts regularly on $B_1 \cup \{\alpha\}$.*

Proof. (a) Since K_1 is 2-transitive on $\Sigma - \{B_1\}$ and has no normal subgroup acting regularly on $\Sigma - \{B_1\}$ it follows from [7, 12.1] that $\bar{K}_1 \cap K_2$ fixes only the blocks B_1 and B_2 setwise and has orbits of equal length in $\Sigma - \{B_1, B_2\}$.

(b) It is easy to show that $\bar{K}_1^\Sigma$ has a trivial centralizer in G_α^Σ, (see [3]). It follows from [6, Lemma 1.4] that the centralizer $C_G(\bar{K}_1)$ of $\bar{K}_1$ acts as a Frobenius group on $B_1 \cup \{\alpha\}$ of degree $1 + b = r^c$ for some prime r and positive integer c, and $C_G(\bar{K}_1)$ has a normal subgroup A such that $A^{B_1 \cup \{\alpha\}}$ is elementary abelian and regular. Moreover, since A_α fixes $B_1 \cup \{\alpha\}$ pointwise, then A_α is a subgroup of $\bar{K}_1$ which centralizes $\bar{K}_1$. Since $\bar{K}_1^\Sigma \simeq \bar{K}_1$ has a trivial centralizer, and since $A_\alpha^\Sigma \simeq A_\alpha$ centralizes $\bar{K}_1^\Sigma$, it follows that A_α is trivial. Thus A is elementary abelian of order r^c.

LEMMA 3. (a) *All long orbits of $\bar{K}_1$ have length $b'(t-1)$, where b' is a divisor of $b = r^c - 1$, and consist of b' points of each block of $\Sigma - \{B_1\}$.*

(b) *The group A fixes some long $\bar{K}_i$-orbit Γ setwise.*

Proof. (a) We show that $K_1 \cap K_2$ is transitive on B_2, and then it follows that its normal subgroup $\bar{K}_1 \cap K_2$ has orbits of equal length, say b', in B_2, for some b' dividing b. Then since $\bar{K}_1$ is transitive on $\Sigma - \{B_1\}$, ([7, 8.8]), it follows that all long orbits of $\bar{K}_1$ have length $b'(t-1)$ and contain b' points of each block of $\Sigma - \{B_1\}$.

Thus we need to show that $K_1 \cap K_2$ is transitive on B_2. Let β be a point of B_2. Since K_2 is transitive on B_2, then $(K_2)_\beta$ has index b in K_2; also since $(K_2)_\beta$ contains $\bar{K}_2$ then $(K_2)_\beta$ is transitive on $\Sigma - \{B_2\}$ and so $(K_2)_\beta \cap K_1 = (K_2 \cap K_1)_\beta$ has index $b(t-1)$ in K_2. It follows that $(K_1 \cap K_2)_\beta$ has index b in $K_1 \cap K_2$, that is, $K_1 \cap K_2$ is transitive on B_2.

(b) Since A centralizes $\bar{K}_1$, it permutes the orbits of $\bar{K}_1$ among themselves. Since A is an r group and since the number of long $\bar{K}_1$-orbits, namely, b/b' is not divisible by r, we conclude that A must fix one of these orbits, say Γ, setwise.

LEMMA 4. *If B_i and B_j are (not necessarily distinct) blocks of $\Sigma - \{B_1\}$ and if a is a nonidentity element of A, then $| B_i^a \cap B_j | \leqslant 1$.*

Proof. Suppose to the contrary that for some $i \geqslant 2, j \geqslant 2$ and for some nonidentity element a of A, we have $| B_i^a \cap B_j | \geqslant 2$. Then $(B_i \cup \{\alpha\})^a$ and $B_j \cup \{\alpha\}$ are blocks of a design with $\lambda = 1$ which have at least two points in common, and hence $(B_i \cup \{\alpha\})^a = B_j \cup \{\alpha\}$. Thus α^a belongs to $B_j \cup \{\alpha\}$, which is a contradiction, for α^a is a point of B_1 since A acts regularly on $B_1 \cup \{\alpha\}$.

LEMMA 5. $b' \neq 1$.

Proof. Suppose that $b' = 1$ and let Γ be a long orbit of $\bar{K}_1$ fixed setwise by A (by Lemma 3). By Lemma 2, $\bar{K}_1 \cap K_2$ fixes only the blocks B_1 and B_2 of Σ setwise. Hence $\bar{K}_1 \cap K_2$ fixes exactly one point of Γ, namely the point γ where $\Gamma \cap B_2 = \{\gamma\}$. Since A centralizes $\bar{K}_1 \cap K_2$ then A fixes γ, and since $\bar{K}_1$ is transitive on $\Sigma - \{B_1\}$ we conclude that A fixes Γ pointwise.

Since $b' = 1$, there is an orbit Γ' of $\bar{K}_1$ of length $t - 1$ distinct from Γ. Let A' be the setwise stabilizer in A of Γ'; the index of A' in A is at most the number of long $\bar{K}_1$-orbits so that $| A : A' | \leqslant b < r^c = | A |$, that is, A' is nontrivial. Let a be a nonidentity element of A'. Then, as in the previous paragraph, we can show that a fixes Γ' pointwise. Thus $B_2^a \cap B_2$ contains $(\Gamma \cap B_2) \cup (\Gamma' \cap B_2)$, a contradiction to Lemma 4. Hence $b' \neq 1$.

LEMMA 6. *The group A is semiregular on Γ, and if γ is a point of Γ, then $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ is a union of x orbits of A and hence contains xr^c points for some $x \geqslant 1$.*

Proof. The group A acts faithfully on Γ; for suppose that a nonidentity element a of A fixed Γ pointwise. Then $B_2^a \cap B_2 \supseteq \Gamma \cap B_2$, which contradicts Lemmas 4 and 5.

Then it follows from [3] that A is semiregular on Γ and $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ is a union of say x orbits of A for any γ in Γ. Thus $|\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma| = xr^c$.

LEMMA 7. (a) *$t-1$ is divisible by r^c.*

(b) *There are xr^c/m blocks of $\Sigma - \{B_1\}$ which contain a point of* $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ *where $\gamma \in \Gamma \cap B_2$ and each of these blocks contains exactly m points of* $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$, *for some positive integer m dividing x.*

Proof. (a) By [7, 3.6], the normalizer N of $(\bar{K}_1)_\gamma$ in $\bar{K}_1$ is transitive on $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ and hence xr^c divides $|N: (\bar{K}_1)_\gamma|$ which divides $|\bar{K}_1: (\bar{K}_1)_\gamma| = b'(t-1)$. Then since b' divides $b = r^c - 1$ it follows that r^c divides $t-1$.

(b) Now $\{\Gamma \cap B_j \mid j \geqslant 2\}$ is a set of blocks of imprimitivity for $(\bar{K}_1)^\Gamma$, and since N is transitive on $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ clearly N is transitive on

$$X = \{\Gamma \cap B_j \mid B_j \cap \mathrm{fix}_\Gamma(\bar{K}_1)_\gamma \neq \phi\}.$$

It follows that each $\Gamma \cap B_j$ in X contains the same number m of points of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$, and hence, that X has $|\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma|/m = xr^c/m$ members. Finally, $\bar{K}_1 \cap K_2$ is transitive on $\Gamma \cap B_2$ and the stabilizer of the point γ of $\Gamma \cap B_2$ is $(\bar{K}_1)_\gamma$. Moreover, $(\bar{K}_1)_\gamma$ fixes exactly m points of $\Gamma \cap B_2$ as we have just shown, and so by [7, 3.6], the normalizer $N \cap K_2$ of $(\bar{K}_1)_\gamma$ in $\bar{K}_1 \cap K_2$ is transitive on $\mathrm{fix}_{\Gamma \cap B_2}(\bar{K}_1)_\gamma$. Hence m divides $|N \cap K_2: (\bar{K}_1)_\gamma|$ which divides

$$|(\bar{K}_1 \cap K_2): (\bar{K}_1)_\gamma| = |\Gamma \cap B_2| = b'.$$

Thus m divides b', and since xr^c/m is an integer then m must divide x.

LEMMA 8. (a) *The orbits of $\bar{K}_1 \cap K_2$ in $\Sigma - \{B_1, B_2\}$ all have length b'.*

(b) *For a point γ of $\Gamma \cap B_2$, $(\bar{K}_1)_\gamma$ fixes my blocks of $\Sigma - \{B_1, B_2\}$ setwise, and each of these blocks contains a point of* $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$.

Proof. Let a be a nonidentity element of A. Then since $\Gamma \cap B_2$ is an orbit of $\bar{K}_1 \cap K_2$ and since A centralizes $\bar{K}_1 \cap K_2$, then $(\Gamma \cap B_2)^a$ is also an orbit of $\bar{K}_1 \cap K_2$. By Lemma 4, $(\Gamma \cap B_2)^a$ consists of one point from each of b' blocks of Σ. Thus $(\Gamma \cap B_2)^a$ corresponds to an orbit Δ of $\bar{K}_1 \cap K_2$ of length b' in $\Sigma - \{B_1, B_2\}$. Hence by Lemma 2, all orbits of $\bar{K}_1 \cap K_2$ in $\Sigma - \{B_1, B_2\}$ have length b'.

Now by Lemma 7, $(\bar{K}_1)_\gamma$ fixes m points of $\Gamma \cap B_2$ and permutes the remaining points nontrivially. Hence, since A centralizes $(\bar{K}_1)_\gamma$, $(\bar{K}_1)_\gamma$ fixes exactly m points of $(\Gamma \cap B_2)^a$ and permutes the remaining points nontrivially. It follows that there are exactly m blocks in the corresponding $(\bar{K}_1 \cap K_2)$-orbit Δ in Σ which contain a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ and the remaining blocks of Δ are permuted nontrivially by $(\bar{K}_1)_\gamma$.

To complete the proof of Lemma 8 we show: if B_j is an arbitrary block of $\Sigma - \{B_1, B_2\}$ which is fixed setwise by $(\bar{K}_1)_\gamma$, then

(a) the $(\bar{K}_1 \cap K_2)$ — orbit Δ' in Σ containing B_j contains exactly m blocks which contain a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$, and the remaining blocks of Δ' are permuted nontrivially by $(\bar{K}_1)_\gamma$, and hence,

(b) B_j contains a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$. To show this, let B_i be a block of Δ containing a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ (where Δ was defined above), and let B_j be an arbitrary block of $\Sigma - \{B_1, B_2\}$ fixed setwise by $(\bar{K}_1)_\gamma$. Then B_i and B_j lie in orbits of $\bar{K}_1 \cap K_2$ in Σ of length b', and hence the groups $\bar{K}_1 \cap K_2 \cap K_i$ and $\bar{K}_1 \cap K_2 \cap K_j$ have index b' in $\bar{K}_1 \cap K_2$. Since $(\bar{K}_1)_\gamma$ fixes B_i and B_j setwise it is a subgroup of $\bar{K}_1 \cap K_2 \cap K_i$ and $\bar{K}_1 \cap K_2 \cap K_j$. Then since $(\bar{K}_1)_\gamma$ has index b' in $\bar{K}_1 \cap K_2$ it follows that $(\bar{K}_1)_\gamma = \bar{K}_1 \cap K_2 \cap K_i = \bar{K}_1 \cap K_2 \cap K_j$. Since $K_1 \cap K_2$ is transitive on $\Sigma - \{B_1, B_2\}$ there is an element g in $K_1 \cap K_2$ such that $B_i^g = B_j$. Hence $(\bar{K}_1)_\gamma^g = (\bar{K}_1 \cap K_2 \cap K_i)^g = \bar{K}_1 \cap K_2 \cap K_j = (\bar{K}_1)_\gamma$, that is, g normalizes $(\bar{K}_1)_\gamma$. It follows that the orbit of $\bar{K}_1 \cap K_2$ in Σ containing B_j, namely Δ^g, contains exactly m blocks which contain a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$, and the remaining blocks are permuted nontrivially by $(\bar{K}_1)_\gamma$.

We now complete the proof of the theorem. By Lemmas 7 and 8 it follows that the number of blocks of $\Sigma - \{B_1\}$ which contain a point of $\mathrm{fix}_\Gamma(\bar{K}_1)_\gamma$ is, on the one hand, xr^c/m, and on the other, $1 + my$. Hence $1 + my = xr^c/m \equiv O \pmod{r^c}$.

Consider the set $X = \{\bar{K}_1 \cap K_2 \cap K_j \mid j > 2\}$. This is a conjugacy class of subgroups of $K_1 \cap K_2$. Since $K_1 \cap K_2$ is transitive on $\Sigma - \{B_1, B_2\}$, then each group in X fixes the same number of blocks of $\Sigma - \{B_1, B_2\}$ setwise. We showed in the proof of Lemma 8 that $(\bar{K}_1)_\gamma$ belongs to X and fixes exactly my blocks of $\Sigma - \{B_1, B_2\}$ setwise. Hence $|X| = (t-2)/my$.

On the other hand, $K_1 \cap K_2$ fixes B_2 as a set and is transitive on it. Hence, since the group $(\bar{K}_1)_\gamma$ of X fixes points of B_2, it follows that each group in X fixes the same number, say z, of points of B_2. Thus $b = z \mid X \mid$, and so $my = (t-2)z/b$. Hence we have $1 + my = 1 + (t-2)z/b \equiv 0 \pmod{r^c}$, and so $b + (t-2)z \equiv 0 \pmod{r^c}$. By Lemma 7, $t - 2 \equiv -1 \pmod{r^c}$, and so $b \equiv z \pmod{r^c}$. However $z \leqslant b < r^c$. Hence $b = z$, and so $my = t - 2$. This means that $(\bar{K}_1)_\gamma$ fixes all blocks of Σ setwise, and hence that $\bar{K}_1 \cap K_2$ acts semiregularly on $\Sigma - \{B_1, B_2\}$. Thus $\bar{K}_1$ acts on $\Sigma - \{B_1\}$ as a Frobenius group. By [7, 5.1], $\bar{K}_1$ has a characteristic subgroup acting regularly on $\Sigma - \{B_1\}$, a contradiction to our assumption that K_1 has no normal subgroup acting regularly on $\Sigma - \{B_1\}$. This completes the proof of the theorem.

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